

Cubic Irrationals and Periodicity via a Family of Multi-dimensional Continued Fraction Algorithms*

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Abstract

We construct a countable family of multi-dimensional continued fraction algorithms, built out of five specific multidimensional continued fractions, and show a real number α is a cubic irrational precisely when its multidimensional continued fraction expansion with respect to at least one element of the countable family is eventually periodic. We interpret this result as the construction of a matrix with entries of non-negative integers such that at least one of the rows is eventually periodic if and only if α is a cubic irrational. This result is built on a careful technical analysis of certain units in cubic number fields and our family of multi-dimensional continued fractions.

1 Introduction

A real number has an eventually periodic decimal expansion precisely when it is rational. A real number has an eventually periodic continued fraction expansion precisely when it is a quadratic irrational. A natural question (which Hermite [10] asked Jacobi in 1839) is if there is a way to represent real numbers as sequences of nonnegative integers such that a number's algebraic properties are revealed by the periodicity of its sequence. Specifically, Hermite wanted an algorithm that returns an eventually periodic sequence of integers if and only if its input is a cubic irrational.

There have been many attempts to produce such an algorithm. These attempts fall into two classes [4]. The first is a class of algorithms based on number geometric interpretations of the standard continued fractions algorithm, which goes back at least to the work of Minkowski. More current examples of such work can be found in [7], [8, 5]. The second class of algorithms tries to generalize the standard continued fractions algorithm arithmetically. So far, no periodic arithmetic

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algorithm has been found. Algorithms in this class are known as “multidimensional continued fractions.” Descriptions of the most well-known of the Multidimensional Continued Fraction algorithms can be found in [13], [12], [3] and in a recent survey [2].

We suspect that there is no such algorithm. Instead, the best possible solution to the Hermite problem will be in the form of a family of algorithms, meaning that for any arbitrary α , we can produce a β such that the sequence of integers associated with (α, β) will be periodic with respect to some algorithm in the family if and only if α is a cubic irrational. Such a family is capable of being encoded as a matrix (a_{ij}) , with $1 \leq i, j < \infty$. Each row will be the sequence of integers associated to some algorithm in the family. In this language, we would want one of the rows to be eventually periodic if and only if α is a cubic irrational. We will construct such a matrix in this paper.

Heuristically, the reason, in part, that a real number α is a quadratic irrational is that the vector $(1, \alpha)$ is an eigenvector of a 2×2 invertible matrix with rational entries that is not a multiple of the identity. More precisely, there needs to be a matrix $A \in GL(2, \mathbb{Q})$ such that $(1, \alpha)A$ is an eigenvector of a 2×2 invertible matrix with rational entries that is not a multiple of the identity. The continued fraction algorithm can be interpreted as a procedure for producing a sequence of matrices in $SL(2, \mathbb{Z})$ (and hence in $GL(2, \mathbb{Q})$) so that we are guaranteed that if α is a quadratic irrational, then there is a matrix A in our sequence with the $(1, \alpha)A$ our desired type of eigenvector. Of course, continued fractions have many other key properties, especially about questions involving Diophantine approximations.

In a similar fashion, numbers α and β will be in the same cubic number field if there is matrix $A \in GL(3, \mathbb{Q})$ such that $(1, \alpha, \beta)A$ is an eigenvector of a 3×3 invertible matrix with rational entries that is not a multiple of the identity. All multidimensional continued fraction algorithms produce sequences of matrices in $GL(3, \mathbb{Q})$ (often in $SL(3, \mathbb{Z})$). Periodicity of the given algorithm corresponds to finding our desired matrix A . It appears that each of the existing algorithms are, in some sense, one-dimensional, and for dimension reasons, not guaranteed to find A . This supports the observation that for almost all existing multidimensional continued fraction algorithms, eventual periodicity means that α and β are in the same cubic number field, but that for none of these algorithms can the converse be shown. This also suggests why we turn to a five dimensional family of multidimensional continued fraction algorithms.

In section 2 we review the family of 216 TRIP maps, from [6]. In section 3 we present the main technical results of our paper. We find a family of Multidimensional Continued Fraction algorithms formed by compositions of 5 of the TRIP maps. We show that, for every cubic number field, there exists a pair (u, u') with u a unit in the cubic number field (or possibly the quadratic extension of the cubic number field by the square root of the discriminant) such that (u, u') has a periodic multidimensional continued fraction expansion under one of the maps in this family of 5 maps. This is the technical heart of the paper, linking periodicity and units. In section 4 we see how to solve the Hermite problem via the periodicity of a row of a matrix. It is here that we show that our countable family of multi-dimensional continued fractions has the property that a real α is a cubic irrational if and only if its multidimensional continued fraction expansion with respect one of the algorithms in the family is eventually periodic.

2 The TRIP Algorithms

This section closely follows sections 2 and 3 from [6].

We begin by describing the original triangle map, which is a multidimensional continued fraction algorithm, as defined in [9] and further developed in [1], [14]. We then introduce permutations into the definition of the triangle map, thereby generating a family of 216 multidimensional continued fractions. Finally, we show how to produce triangle sequences, which are the analog of continued fraction expansions.

2.1 The Triangle Map

The ordinary continued fraction expansion is computed by iterating the Gauss map on the unit interval. The triangle map generalizes this method. Instead of the unit interval, we use a 2-simplex, i.e. a triangle. We think of this triangle as lying in \mathbb{R}^3 . Specifically, define

$$\Delta = \{(1, x, y) : 1 \geq x \geq y > 0\}$$

Define $\pi : \mathbb{R}^3 - \{(0, x, y)\} \rightarrow \mathbb{R}^2$ by setting

$$\pi(z, x, y) = \left(\frac{x}{z}, \frac{y}{z}\right).$$

The vectors

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

are the vertices of Δ . Now in order to partition Δ , we consider the following two matrices:

$$A_0 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}, A_1 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Let $B = (\mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{v}_3)$. Then the column vectors of BA_0 and BA_1 describe a disjoint partition of Δ . We iterate this division process, and define Δ_k to be the image of Δ under $A_1^k A_0$. Then let T be a bijective map from $\pi(\Delta_k)$ to $\pi(\Delta)$ given by

$$\begin{aligned} T(x, y) &= \pi \left[(1, x, y) \cdot (BA_0^{-1} A_1^{-k} B^{-1})^T \right] \\ &= \left(\frac{y}{x}, \frac{1 - x - ky}{x} \right) \end{aligned}$$

where $k = \lfloor \frac{1-x}{y} \rfloor$. This map T , which is called the **triangle map**, is analogous to the Gauss map.

2.2 Incorporating Permutations

The triangle map consists of a process of partitioning of a triangle with vertices $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ into the triangles with vertices $(\mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_1 + \mathbf{v}_3)$ and $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_1 + \mathbf{v}_3)$. The essential thing to note is that this process assigns a particular ordering of vertices to both the vertices of the original triangle and the vertices of the two triangles produced. But this ordering is by no means canonical. We can permute the vertices at several stages of the triangle division. This leads to the following definition.

Definition 1. For every $(\sigma, \tau_0, \tau_1) \in S_3^3$, define

$$F_0 = \sigma A_0 \tau_0 \text{ and } F_1 = \sigma A_1 \tau_1$$

by thinking of σ , τ_0 , and τ_1 as column permutation matrices.

Given any $(\sigma, \tau_0, \tau_1) \in S_3^3$, we can partition Δ in a distinct way using the matrices F_0 and F_1 instead of A_0 and A_1 . This leads to the definition of a family of multidimensional continued fractions algorithms, each specified by a $(\sigma, \tau_0, \tau_1) \in S_3^3$. Because $|S_3^3| = 216$, this family has 216 elements.

Most important for our present purposes, we can define the map analogous to the Gauss map for any of the 216 multidimensional continued fractions algorithms.

Definition 2. Given any $(\sigma, \tau_0, \tau_1) \in S_3^3$, define

$$T_{\sigma, \tau_0, \tau_1}(1, x, y) = \pi \left[(1, x, y) \cdot (BF_0^{-1}F_1^{-k}B^{-1})^T \right]$$

Recall that $B = (\mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{v}_3)$ and that $\pi(a, b, c) = (1, \frac{b}{a}, \frac{c}{a})$.

Note that the matrix $(BF_0^{-1}F_1^{-k}B^{-1})^T$ is in $SL(3, \mathbb{Z})$. The triangle partition maps $T_{\sigma, \tau_0, \tau_1}^k$ are called **TRIP Maps**.

2.3 TRIP Sequences

We now use this to define a method of constructing an integer sequence. The sequence is analogous to the continued fraction expansion of a number.

Definition 3. Given $(x, y) \in \pi(\Delta)$, we recursively define a_n to be the non-negative integer such that $T_{\sigma, \tau_0, \tau_1}^{a_n}(x, y)$ is in $\pi(\Delta_{a_n})$. The **triangle sequence** of (x, y) with respect to (σ, τ_0, τ_1) is (a_0, a_1, \dots) .

2.4 An Even Larger Family: COMBO TRIP Maps

Finally, and most importantly, we can obtain a much larger family of algorithms by mixing and matching these maps. For example, we can carry out the first subdivision of Δ using (σ, τ_0, τ_1) , the second subdivision using $(\sigma', \tau'_0, \tau'_1)$, and so forth. As we shown in [6], many well known multidimensional continued fractions are compositions of a finite number of the 216 permutation-division maps. To each of these COMBO TRIP maps and to each (x, y) is associated a sequence of non-negative integers in analog to the usual TRIP sequence. Again, all of this is in [6].

2.5 Periodicity

We would like to understand properties of periodicity of TRIP sequences. In order to investigate periodicity of these sequences, it turns out to be much easier to work with eigenvectors of the matrices representing the T that produces the TRIP sequence. This yields the following important proposition, which will prove indispensable to us in this paper.

Proposition 1. Suppose T is a composition of a finite number of TRIP maps of the form $T_{e, e, e}$, $T_{e, (23), e}$, $T_{e, (123), (123)}$, $T_{(23), (23), e}$, or $T_{(13), (12), e}$. If $(1, \alpha, \beta) \in \Delta$ is an eigenvector of the matrix representation of T , then (α, β) has a purely periodic triangle sequence of period 1 with respect to T .

The proof is in sections six and seven of [6]. The key idea is that, if $(1, \alpha, \beta)$ is an eigenvector of the matrix representing T , then $T(\alpha, \beta) = (\alpha, \beta)$.

3 A Family of Algorithms Yielding Periodicity

We are now ready to present the main technical results of this paper. We begin by introducing the countable family of multidimensional continued fraction algorithms that we will use.

3.1 A Countable Family of Algorithms

The family of algorithms we consider is constructed from five TRIP Maps, namely from maps

$$T_{e,e,e} \quad T_{e,(23),e} \quad T_{e,(123),(123)} \quad T_{(23),(23),e} \quad T_{(13),(12),e}$$

From these we create the following three classes of maps, where $n, k \in \mathbb{Z}_{\geq 0}$:

1. $(T_{e,(123),(123)}^1)^n \circ T_{e,e,e}^k$
2. $(T_{e,(123),(123)}^1)^n \circ T_{e,(23),e}^k$
3. $(T_{(13),(12),e}^0)^n \circ T_{(23),(23),e}^k$

From now on, we refer to these as Class 1, Class 2, and Class 3. We can describe these maps as algorithms in the following manner. We have from definition that

$$T_{e,(123),(123)}(x, y) = \left(\frac{x}{1 - kx}, \frac{y}{1 - ky} \right) \text{ where } k = \lfloor \frac{1-x}{x} \rfloor$$

$$T_{e,e,e}(x, y) = \left(\frac{y}{x}, \frac{1-x-ky}{x} \right) \text{ where } k = \lfloor \frac{1-x}{y} \rfloor$$

$$T_{e,(23),e}(x, y) = \left(\frac{y}{x}, \frac{(k+1)x + y - 1}{x} \right) \text{ where } k = \lceil \frac{1}{x} \rceil - 1$$

$$T_{(13),(12),e}(x, y) = \left(\frac{x}{1 - ky}, \frac{y}{1 - ky} \right) \text{ where } k = \lfloor \frac{1-x}{y} \rfloor$$

and $T_{(23),(23),e}(x, y)$ has first coordinate

$$\frac{(1 + \frac{1}{2}(-1 + (-1)^k))x - (-1)^k y}{x},$$

and second coordinate

$$\frac{-1 + (2 + \frac{1}{2}(-1 + (-1)^k) + \frac{1}{4}(1 - (-1)^k + 2k))x + (-(-1)^k + \frac{1}{2}(-1 + (-1)^k))y}{x},$$

where $k = \lfloor \frac{2-2x}{x} \rfloor$. Thus we can in principle compute the maps for each of the three classes of COMBO TRIP maps.

3.2 Main Results on Units

We now present the main technical underpinning for this paper, followed by an important collorary.

Theorem 1. *Let K be a cubic number field. If $u \in \mathcal{O}_K$ is a unit such that $0 < u < 1$, then either (u, u^2) , (u^2, u^4) , $(u, u^2 - u)$, $(u^2, u^2 - u^4)$, $(uu', (uu')^2 - uu')$, or $((uu')^2, (uu')^2 - (uu')^4)$, where u' is a conjugate of u , has a periodic triangle sequence under a map in Class 1, 2, or 3.*

Now, by Dirichlet's Unit Theorem, every cubic number field contains an infinite number of units in the interval $(0, 1)$. This means there are an infinite number of these ordered pairs. This yields the following important corollary.

Corollary 1. *Let K be a cubic number field, u be a real unit in \mathcal{O}_K , with $0 < u < 1$ and $E = K(\sqrt{\Delta_{\mathbb{Q}(u)}})$ (where $\Delta_{\mathbb{Q}(u)}$ is the discriminant of $\mathbb{Q}(u)$). Then there exists a point (α, β) , with $\alpha, \beta \in E$ irrational, such that (α, β) has a periodic TRIP sequence.*

This theorem and corollary form the essence of this paper. The proof of the theorem reduces to calculations, which, though not difficult, requires a number of rather technical lemmas, which are stated below.

Lemma 1. *Let K be a cubic number field. Let u be a unit in \mathcal{O}_K . Then there exists $A, B \in \mathbb{Z}$ such that either $u^3 + Au^2 + Bu + 1 = 0$ or $u^3 + Au^2 + Bu - 1 = 0$.*

Proof. Since we know u is an algebraic integer, there must exists integers A, B, C such that $u^3 + Au^2 + Bu + C = 0$. This means $Cu^{-3} + Bu^{-2} + Au^{-1} + 1 = 0$. Since u^{-1} must also be an algebraic integer, we know $C = \pm 1$. This, implies $u^3 + Au^2 + Bu + 1 = 0$ or $u^3 + Au^2 + Bu - 1 = 0$. □

We now use Lemma 1 in order to prove Lemma 2.

Lemma 2. *Let $\alpha \in \mathbb{R}$ be a cubic irrational. Let u be a unit in $\mathcal{O}_{\mathbb{Q}(\alpha)}$ and Δ be the discriminant of the cubic number field $\mathbb{Q}(\alpha)$. There exist $P, Q \in \mathbb{Z}_{>0}$ such that some element, strictly between zero and one and in $\mathbb{Q}(\alpha, \sqrt{\Delta})$, satisfies one of the following equations:*

1. $x^3 + Px^2 + Qx - 1 = 0$
2. $x^3 - Px^2 - Qx + 1 = 0$
3. $x^3 + Qx - 1 = 0$
4. $x^3 - Px^2 + Qx - 1 = 0$ where $Q > P$.

Proof. Let $0 < u < 1$ be an irrational unit in $\mathbb{Q}(\alpha)$. From Lemma 1 we know that there must exist $A, B \in \mathbb{Z}_{\geq 0}$ such that one of $f_{\delta_1, \delta_2, \delta_3}(u) = u^3 \pm Au^2 \pm Bu \pm 1 = 0$ must hold, where δ_1, δ_2 , and δ_3 specify the signs in front of A, B , and 1. Let $f_{\epsilon_1, \epsilon_2, \epsilon_3}$ be a specific one of these polynomials such that $f_{\epsilon_1, \epsilon_2, \epsilon_3}(u) = 0$. This yields 18 cases.

We first consider the cases when $A, B \neq 0$.

Case 1. $\epsilon_1 = +, \epsilon_2 = +, \epsilon_3 = +$

In this case, $f_{\epsilon_1, \epsilon_2, \epsilon_3}(u) = u^3 + Au^2 + Bu + 1$. Since this polynomial cannot have any roots between 0 and 1, this yields a contradiction, and so this case cannot occur.

Case 2. $\epsilon_1 = +, \epsilon_2 = +, \epsilon_3 = -$

This means $f_{\epsilon_1, \epsilon_2, \epsilon_3}$ is in the form of (1) and so we are done.

Case 3. $\epsilon_1 = +, \epsilon_2 = -, \epsilon_3 = +$

This means $f_{\epsilon_1, \epsilon_2, \epsilon_3}(u) = u^3 + Au^2 - Bu + 1 = 0$, which implies $u(u^2 + Au - B) = -1$. This means

$$\begin{aligned} u^2 &= \frac{1}{u^4 + A^2u^2 + B^2 + 2Au^3 - 2Bu^2 - 2ABu} \\ &= \frac{1}{u^4 + (A^2 - 2B)u^2 + B^2 + 2A(u^3 - Bu)} \\ &= \frac{1}{u^4 + (A^2 - 2B)u^2 + B^2 + 2A(-1 - Au^2)} \\ &= \frac{1}{u^4 + (-A^2 - 2B)u^2 + B^2 - 2A} \end{aligned}$$

Setting $v = u^2$, we have

$$v = \frac{1}{v^2 + (-A^2 - 2B)v + B^2 - 2A}$$

This implies $v^3 + (-A^2 - 2B)v^2 + (B^2 - 2A)v - 1 = 0$. Since we know $0 < u < 1$, this means $0 < v < 1$ too. So for $v^3 + (-A^2 - 2B)v^2 + (B^2 - 2A)v - 1 = 0$ to have a solution between 0 and 1 we need $(B^2 - 2A) > 0$. Letting $P = A^2 + 2B$ and $Q = B^2 - 2A$, we are left with $v^3 - Pv^2 + Qv - 1 = 0$ where $P, Q \in \mathbb{Z}^+$. This reduces to Case 6.

Case 4. $\epsilon_1 = +, \epsilon_2 = -, \epsilon_3 = -$

This means $u^3 + Au^2 - Bu - 1 = 0$, so $u(u^2 + Au - B) = 1$. As before, we have

$$\begin{aligned} u^2 &= \frac{1}{u^4 + A^2u^2 + B^2 + 2Au^3 - 2Bu^2 - 2ABu} \\ &= \frac{1}{u^4 + (A^2 - 2B)u^2 + B^2 + 2A(u^3 - Bu)} \\ &= \frac{1}{u^4 + (A^2 - 2B)u^2 + B^2 + 2A(1 - Au^2)} \\ &= \frac{1}{u^4 + (-A^2 - 2B)u^2 + B^2 + 2A} \end{aligned}$$

which then implies, setting $v = u^2$, that $v^3 + (-A^2 - 2B)v^2 + (B^2 + 2A)v - 1 = 0$. This reduces to Case 6.

Case 5. $\epsilon_1 = -, \epsilon_2 = +, \epsilon_3 = +$

This means $u^3 - Au^2 + Bu + 1 = 0$ and so $u(u^2 - Au + B) = -1$. As before, we have

$$\begin{aligned} u^2 &= \frac{1}{u^4 + A^2u^2 + B^2 - 2Au^3 + 2Bu^2 - 2ABu} \\ &= \frac{1}{u^4 + (A^2 + 2B)u^2 + B^2 - 2A(u^3 + Bu)} \\ &= \frac{1}{u^4 + (A^2 + 2B)u^2 + B^2 - 2A(-1 + Au^2)} \\ &= \frac{1}{u^4 + (2B - A^2)u^2 + B^2 + 2A} \end{aligned}$$

which then implies, setting $v = u^2$, that $v^3 + (2B - A^2)v^2 + (B^2 + 2A)v - 1 = 0$. If $2B - A^2 \geq 0$, then $g_{\gamma_1, \gamma_2, \gamma_3}$ is of desired forms (1). If $2B - A^2 < 0$, then, as in the previous case, $g_{\gamma_1, \gamma_2, \gamma_3}$ must be of desired form (4).

Case 6. $\epsilon_1 = -, \epsilon_2 = +, \epsilon_3 = -$

It is here where we will see our need to consider units not in just the original cubic number field but possibly in a quadratic extension. We have that u is a real unit between zero and one that is a root of the irreducible polynomial

$$f(x) = x^3 - Ax^2 + Bx - 1.$$

We consider the three subcases of $A = B$, $A < B$ and $A > B$. If $A = B$, then $f(x)$ is not irreducible, since we have

$$x^3 - Ax^2 + Ax - 1 = (x - 1)(x^2 - (A - 1)x + 1).$$

Hence this cannot happen.

If $B > A$, $f_{\epsilon_1, \epsilon_2, \epsilon_3}$ is of desired form (4) and we are done.

This leaves the case for when $A > B$. Label the three roots of $f(x)$ by our original u , and u_2 and u_3 . We first will show that one of these two other roots is also real, between zero and one. Note that $f(0) = -1 < 0$ and

$$f(1) = 1 - A + B - 1 < 0.$$

(This is the step where we are using that $A > B$.) Since $f(u) = 0$ and since u is not a double root (using here that $f(x)$ is irreducible), we can indeed assume that u_2 is real between zero and one. We know that

$$uu_2u_3 = 1,$$

meaning that u_3 is real and greater than one. We know that $1/u_3 = uu_2$ is real between zero and one and will have minimal polynomial

$$x^3 - Bx^2 + Ax - 1,$$

reducing our problem to the subcase where $B > A$. Also, note that even with u being a unit in a cubic number field K , we can only claim that u_3 and hence $1/u_3$ are units in the quadratic extension $K(\sqrt{\Delta_{\mathbb{Q}(u)}})$

Case 7. $\epsilon_1 = -, \epsilon_2 = -, \epsilon_3 = +$

If $u^3 - Au^2 - Bu + 1 = 0$, then $f_{\epsilon_1, \epsilon_2, \epsilon_3}$ is of desired form (2).

Case 8. $\epsilon_1 = -, \epsilon_2 = -, \epsilon_3 = -$

The equation $u^3 - Au^2 - Bu - 1 = 0$ will not have any solutions between 0 and 1 so this case cannot occur.

We now consider the cases when $B = 0$.

Case 9. $\epsilon_1 = +, B = 0, \epsilon_3 = +$

The equation $u^3 + Au^2 + 1 = 0$ will not have any solutions between 0 and 1 so this case cannot occur.

Case 10. $\epsilon_1 = +, B = 0, \epsilon_3 = -$

This means $u^3 + Au^2 - 1 = 0$, which implies $u(u^2 + Au) = 1$. This means

$$\begin{aligned} u^2 &= \frac{1}{u^4 + A^2u^2 + 2Au^3} \\ &= \frac{1}{u^4 + A^2u^2 + 2A(1 - Au^2)} \\ &= \frac{1}{u^4 - A^2u^2 + 2A}. \end{aligned}$$

As before, set $v = u^2$, which means that $v^3 - A^2v^2 + 2Av - 1 = 0$. We are now in case 6.

Case 11. $\epsilon_1 = -, B = 0, \epsilon_3 = +$

If $u^3 - Au^2 + 1 = 0$, then $u(u^2 - Au) = -1$ and so

$$\begin{aligned} u^2 &= \frac{1}{u^4 + A^2u^2 - 2Au^3} \\ &= \frac{1}{u^4 + A^2u^2 - 2A(-1 + Au^2)} \\ &= \frac{1}{u^4 - A^2u^2 + 2A}. \end{aligned}$$

As in the previous cases, letting $v = u^2$, yields the equation $v^3 - A^2v^2 + 2Av - 1 = 0$, which reduces to Case 6.

Case 12. $\epsilon_1 = -, B = 0, \epsilon_3 = -$

The equation $u^3 - Au^2 - 1 = 0$ has no solutions between 0 and 1, so this case cannot occur.

Next, we consider the cases when $A = 0$.

Case 13. $A = 0, \epsilon_2 = +, \epsilon_3 = +$

The equation $u^3 + Bu + 1 = 0$ has no solutions between 0 and 1, so this case cannot occur.

Case 14. $A = 0, \epsilon_2 = +, \epsilon_3 = -$

If $u^3 + Bu - 1 = 0$, then $f_{\epsilon_1, \epsilon_2, \epsilon_3}$ is of desired form (3).

Case 15. $A = 0, \epsilon_2 = -, \epsilon_3 = +$

Suppose that $u^3 - Bu + 1 = 0$. Then $u(u^2 - B) = -1$ and so

$$u^2 = \frac{1}{u^4 - 2Bu^2 + B^2}$$

Then, letting $v = u^2$, we have $v^3 - 2Bv^2 + B^2v - 1 = 0$, which is case 6.

Case 16. $A = 0, \epsilon_2 = -, \epsilon_3 = -$

The equation $u^3 - Bu - 1 = 0$ has no solutions between 0 and 1, so this case cannot occur.

Case 17. $A = 0, B = 0, \epsilon_3 = +$

The equation $u^3 - 1 = 0$ has no solutions between 0 and 1, so this case cannot occur.

Case 18. $A = 0, B = 0, \epsilon_3 = -$

The equation $u^3 + 1 = 0$ has no solutions between 0 and 1, so this case cannot occur. \square

Thus, we have now showed that for every cubic number field K , some element u in a quadratic extension of K will be the solution to one of 4 classes of equations. We next want to examine the periodicity properties of solutions of these four equations. Our goal is to be able to use u to produce points that have periodic triangle sequences under a map in one of our three classes. The next few lemmas describe, for the above equations, points and corresponding combinations of maps such that these points have periodic triangle sequences under a combination of these maps.

Lemma 3. *Let $A, B \in \mathbb{Z}$ with $A \geq 0$ and $B \geq 1$. If $\alpha^3 + A\alpha^2 + B\alpha - 1 = 0$, then (α, α^2) has a periodic triangle sequence under a map in Class 1.*

Proof. The matrix representation of $T_{e,(123),(123)}^1$ is

$$T_{e,(123),(123)}^1 = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

Raising this to some integer power B yields

$$(T_{e,(123),(123)}^1)^B = \begin{pmatrix} 1 & 0 & 0 \\ -B & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We know that

$$T_{e,e,e}^A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & -A \end{pmatrix}.$$

The product $(T_{e,(123),(123)}^1)^{(B-1)} \cdot T_{e,e,e}^A$ is

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -B \\ 0 & 1 & -A \end{pmatrix},$$

which has eigenvector $(1, \alpha, \alpha^2)$ where $\alpha^3 + A\alpha^2 + B\alpha - 1 = 0$, as desired. \square

Lemma 4. *Let $A, B \in \mathbb{Z}_{>0}$. Then, if $\alpha^3 - A\alpha^2 - B\alpha + 1 = 0$, (α, α^2) has a periodic triangle sequence under a map in Class 2.*

Proof. We know that the matrix representation of $T_{e,(23),e}^{(A-1)}$ is

$$T_{e,(23),e}^{(A-1)} = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 1 \\ 0 & 1 & A \end{pmatrix}.$$

The product $(T_{e,(123),(123)}^1)^{(B-1)} \cdot T_{e,(23),e}^{(A-1)}$ is

$$\begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & B \\ 0 & 1 & A \end{pmatrix},$$

which has eigenvector $(1, \alpha, \alpha^2)$ where $\alpha^3 - A\alpha^2 - B\alpha + 1 = 0$, as desired. \square

Lemma 5. *Let $A, B \in \mathbb{Z}_{>0}$, with $B > A$. Then, if $\alpha^3 - A\alpha + B\alpha - 1 = 0$, $(\alpha, \alpha - \alpha^2)$ has a periodic triangle sequence under a map in Class 3.*

Proof. The matrix representation of $T_{(13),(12),e}^0$ is

$$T_{(13),(12),e}^0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix},$$

and

$$(T_{(13),(12),e}^0)^A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -A & 0 & 1 \end{pmatrix}.$$

We know that

$$T_{(23),(23),e}^{(2X-4)} = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 1 & X \\ 0 & -1 & -1 \end{pmatrix}.$$

The product $(T_{(13),(12),e}^0)^A \cdot T_{(23),(23),e}^{(2(1+B-A)-4)}$ is

$$\begin{pmatrix} 0 & 0 & -1 \\ 1 & 1 & 1+B-A \\ 0 & -1 & -1+A \end{pmatrix},$$

which has eigenvector $(1, \alpha, \alpha - \alpha^2)$ where $\alpha^3 - A\alpha^2 + B\alpha - 1 = 0$, as desired. \square

We are now ready for the proof of the main theorem of this section, which will now be quite simple.

Proof. Let $u \in \mathcal{O}_K$ be a unit such that $0 < u < 1$. Then by Lemma 2 either $v = u$, $v = u^2$, or $v = uu'$ for some conjugate u' of u is a solution to equations (1), (2), (3), or (4). If v is a solution to (1), (2), (3), then by Lemma 3 or Lemma 4, the triangle sequence of the point (v, v^2) is periodic with respect to a map in Class 1 or Class 2. If v is a solution to (4), then by Lemma 5, the triangle sequence of $(v, v - v^2)$ is periodic with respect to a map in Class 3. \square

4 The TRIP Matrix: Periodicity of a Row and the Hermite Problem

We will now see how the above results present an approach to the Hermite problem. An ideal solution to the Hermite problem would be that we could find a single algorithm T such that given an arbitrary α we could produce a β such that the sequence of integers associated with (α, β) with respect to T is periodic if and only if α is a cubic irrational. However, as discussed in the introduction, it seems as though no such single algorithm can exist.

As mentioned in the introduction, we think that the best possible solution to the Hermite problem is in the form of a family of algorithms, meaning that for any arbitrary α , we can produce a β such that the sequence of integers associated with (α, β) will be periodic with respect to some algorithm in the family if and only if α is a cubic irrational. Such a family is capable of being encoded as a matrix (a_{ij}) , with $1 \leq i, j < \infty$. Each row will be the sequence of integers associated to some algorithm in the family. In this language, we would want one of the rows to be eventually periodic if and only if α is a cubic irrational. Such a matrix will now be constructed.

In this paper we have found that given any cubic irrational α , we are guaranteed that there is a unit u in $\mathbb{Q}(\alpha)$ (or possibly in a quadratic extension) such that (u, u^2) , (u^2, u^4) , $(u, u^2 - u)$, $(u^2, u^2 - u^4)$, $(uu', (uu')^2 - uu')$, or $((uu')^2, (uu')^2 - (uu')^4)$, where u' is a conjugate of u , has a periodic triangle sequence for one of the chosen COMBO TRIP maps. Our class of three TRIP maps, namely the TRIP maps $(T_{e,(123),(123)}^1)^n \circ T_{e,e,e}^k$, $(T_{e,(123),(123)}^1)^n \circ T_{e,(23),e}^k$ and $(T_{(13),(12),e}^0)^n \circ T_{(23),(23),e}^k$, form a countable family of TRIP maps, which for this section we denote by \mathcal{F} .

Suppose we are given a real α . The set of all pairs of real numbers (a, b) of the form

$$(a_1 + a_2\alpha + a_3\alpha^2 + a_4\sqrt{k} + a_5\alpha\sqrt{k} + a_6\alpha^2\sqrt{k}, b_1 + b_2\alpha + b_3\alpha^2 + b_4\sqrt{k} + b_5\alpha\sqrt{k} + b_6\alpha^2\sqrt{k})$$

for $a_1, \dots, a_6, b_1, \dots, b_6 \in \mathbb{Q}$ and k a positive integer, forms a countable set. Let \mathcal{P} be the subset of these pairs (a, b) such that (a, b) is in our triangle $\pi(\Delta) = \{(x, y) : 1 \geq x \geq y > 0\}$ and such that the three numbers $1, a, b$ are linearly independent over \mathbb{Q} .

Pick a bijection h between \mathbb{N} and the countable set $\mathcal{F} \times \mathcal{P}$. Then for each integer n , we have $h(n) = (T(n), (a(n), b(n)))$, where each $T(n)$ is one of our TRIP maps and $(a(n), b(n))$ are pairs of reals in the triangle $\pi(\Delta)$.

Definition 4. The TRIP matrix $\mathcal{T}(\alpha)$ is the matrix with non-negative integer entries whose n th row is the TRIP sequence of $(a(n), b(n))$ with respect to the $T(n)$ -algorithm.

Thus

$$\mathcal{T}(\alpha) = \begin{pmatrix} \text{the } T(1) \text{ sequence for } (a(1), b(1)) \\ \text{the } T(2) \text{ sequence for } (a(2), b(2)) \\ \text{the } T(3) \text{ sequence for } (a(3), b(3)) \\ \vdots \end{pmatrix}$$

Theorem 2. *A real number α is a cubic irrational if and only if at least one of the rows of the TRIP matrix $\mathcal{T}(\alpha)$ is eventually periodic.*

Proof. The forward direction follows from the earlier part of this paper. Namely, we have found that given any cubic irrational α , we are guaranteed that there is a unit u in $\mathbb{Q}(\alpha)$ (or possibly a quadratic extension) such that (u, u^2) , (u^2, u^4) , $(u, u^2 - u)$, $(u^2, u^2 - u^4)$, $(uu', (uu')^2 - uu')$, or $((uu')^2, (uu')^2(-uu')^4)$, where u' is a conjugate of u , has a periodic triangle sequence for one of the chosen COMBO TRIP maps. These pairs are in the set \mathcal{P} . One of our rows must then not only be periodic but purely periodic.

The reverse direction would be hard to prove from scratch but will follow from results in [6], namely from that paper's Theorem 6.2, Proposition 6.3 and Proposition 7.11. \square

In general, for almost all known multi-dimensional continued fractions, it is usually known that periodicity for a given real α implies that α is no worse than a cubic irrational. What has never been shown is the converse. This is reflected in that the meat of this paper is in showing that given a cubic real α we are guaranteed to find an algorithm for which we have periodicity.

5 Questions

A lot of questions remain. First, what other families of multi-dimensional continued fractions exist whose associated matrix (in analog to our TRIP matrix $\mathcal{T}(\alpha)$) has a periodic row if and only if α is cubic? Is there there a way to stay in the initial cubic number field $\mathbb{Q}(\alpha)$ and not go the larger number field $\mathbb{Q}(\alpha, \sqrt{\Delta})$? Can we require all the elements in a family of multi-dimensional continued fractions to fall into the call of such maps as in [11] .

Once having chosen our family of TRIP maps, there are then many questions about the TRIP matrix $\mathcal{T}(\alpha)$ which are in direct analog to what is known about continued fraction expansions. These questions strike us as currently accessible.

The matrix $T(\alpha)$ should contain all information about the number α . What other algebraic properties of α can be obtained from $T(\alpha)$?

Of course, the original Hermite problem is still open. In light of this paper, we think the correct line of attack would be in showing that no single multidimensional continued fraction algorithm will have periodicity being equivalent to a number being a cubic. This strikes us as quite hard.

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